

# Orbital phases of fermions in an asymmetric optical ladder

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We study a quantum ladder of interacting fermions with coupled  $s$  and  $p$  orbitals. Such a model describes dipolar molecules or atoms loaded into a double-well optical lattice, dipole moments being aligned by an external field. The two orbital components have distinct hoppings. The tunneling between them is equivalent to a partial Rashba spin-orbital coupling when the orbital space ( $s, p$ ) is identified as spanned by pseudo-spin 1/2 states. A rich phase diagram, including incommensurate orbital density wave, pair density wave and other exotic superconducting phases, is proposed with bosonization analysis. In particular, superconductivity is found in the repulsive regime.

**Introduction.** Orbital degree of freedom [1] plays a fundamental role in understanding the unconventional properties in solid state materials [2]. Recent experiments in optical lattices have demonstrated that orbitals can also be used to construct quantum emulators of exotic models beyond natural crystals. Orbital lattices are attracting growing interests due to their unique and fascinating properties resulting from the spatial nature of the degenerate states. For example, the bosonic  $p_x + ip_y$  superfluid [3–7] state has been prepared on a bipartite square lattice [8], and later the other complex superfluid with  $s$  and  $p$  orbitals correlated was observed on a hexagonal lattice [9].

Previous study on multicomponent cold gases mainly focused on hyperfine states of alkali atoms [10, 11]. In a cold gas of atoms with two approximately degenerate hyperfine states, the realized pseudo-spin SU(2) symmetry makes it possible to emulate Fermi Hubbard model in optical lattices [12–14]. To engineer spin-orbital couplings and the resulting topological phases, one has to induce Raman transitions between the hyperfine states to break the pseudo-spin symmetry [15–17]. In contrast, due to the spatial nature of the orbital degrees of freedom, the symmetry in orbital gases, such as that in  $p_x + ip_y$  superfluid [8], can be controlled by simply changing the lattice geometry [8, 9, 18, 19]. With a certain lattice geometry, a spin-orbital like coupling can naturally appear in an orbital gas with  $s$  and  $p$ -orbitals without Raman transitions [20]. Theoretical studies of orbital physics largely focusing on two or three dimensions suggest exotic orbital phases [3–6, 21–29] beyond the scope of spin physics.

In this article, we study a one dimensional orbital ladder with  $s$  and  $p$  orbitals coupled [20, 30]. We shall derive such an effective model for dipolar molecules or atoms loaded in a double-well optical lattice. The tunneling rates (or effective mass) of each orbital component are highly tunable by changing the lattice strength. The coupling between  $s$  and  $p$  orbitals mimics the spin-orbital couplings [20]. A rich phase diagram, includ-

ing incommensurate orbital density wave (ODW), pair density wave (PDW) [31–33], and other exotic superconducting phases, is found with bosonization analysis. The PDW phase realized here is a superconducting phase, that features an oscillating Cooper pair field with a period of  $\pi$ . The incommensurate ODW phase has an oscillating particle-hole pair, which tends to break the time-reversal symmetry. An exotic superconducting phase on the repulsive side is also discovered.

**Model.** Consider a cold ensemble of polar molecules or atoms, e.g.  $^{40}\text{K}^{87}\text{Rb}$  [34–36],  $^{23}\text{Na}^{40}\text{K}$  [37] or Dy [38], whose dipole moments are controlled by an external field as demonstrated in experiments. Let the ensemble trapped by a ladder-like optical lattice of the type studied in [20]. That is, the lattice consists of two chains of potentials of unequal depth, with the molecules or atoms residing on the  $s$  and  $p$  orbital levels of the shallow and deep chains, respectively, in the tight binding regime. Recent experiments have reported various lattice configurations of double-well potentials formed by interfering laser beams, showing the unprecedented tunability of the relative depth between two sub-wells [8, 9, 18]. The single particle Hamiltonian of the  $sp$ -orbital ladder is then given as [20]

$$H_0 = \sum_j C_j^\dagger \begin{bmatrix} -t_s & -t_{sp} \\ t_{sp} & t_p \end{bmatrix} C_{j+1} + h.c. \quad (1)$$

where  $C_j^\dagger = [a_s^\dagger(j), a_p^\dagger(j)]$ , and  $a_s^\dagger$  ( $a_p^\dagger$ ) is the creation operator for the  $s$ -orbital ( $p$ -orbital). The lattice constant is set as the length unit in this paper. The band structure is readily obtained by Fourier transform  $C_j = \int \frac{dk}{2\pi} \tilde{C}(k) e^{ikj}$ . The Hamiltonian in the momentum space reads as  $H_0 = \int \frac{dk}{2\pi} \tilde{C}^\dagger(k) \tilde{\mathcal{H}}(k) \tilde{C}(k)$ , with  $\tilde{\mathcal{H}}(k) = h_0(k)\sigma_0 + \vec{h}(k) \cdot \vec{\sigma}$ , where  $h_0(k) = (t_p - t_s)\cos(k)$ ,  $h_x(k) = 0$ ,  $h_y(k) = 2t_{sp}\sin(k)$  and  $h_z(k) = -(t_p + t_s)\cos(k)$ . Here  $\sigma_0$  is the identity matrix and  $\sigma_{x,y,z}$  are Pauli matrices. The two bands are given by  $E_{\pm}(k) = h_0(k) \pm \sqrt{h_y^2(k) + h_z^2(k)}$ , which are shown in FIG. 1. The Hamiltonian is rewritten as  $H_0 = \int \frac{dk}{2\pi} \sum_{\varphi=\pm} E_{\varphi}(k) \phi_{\varphi}^\dagger(k) \phi_{\varphi}(k)$ . We define an angle variable  $\theta$  by  $\cos(\theta(k)) = h_z/|\vec{h}|$  and  $\sin(\theta(k)) = h_y/|\vec{h}|$  to

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save writing. Here, we only consider lower than half filling, i.e., less than one particle per unit cell. The lower band is thus partially filled and the upper band is empty. Since we are interested in the low-energy physics, the spectrum  $E_-$  is linearized around the Fermi momenta  $\pm k_{F\nu}$ . Here,  $\nu = A$  or  $B$ , and  $\pm k_{FA}$  are inner Fermi points and  $\pm k_{FB}$  are outer Fermi points (FIG. 1). The resulting Fermi velocities are  $v_{F\nu} = |\frac{\partial E_-(k)}{\partial k}|_{k=k_{F\nu}}$ . The operators capturing the low energy fluctuations are defined with right ( $\Psi$ ) and left ( $\bar{\Psi}$ ) moving modes  $\Psi_A(k) = \phi_-(k_{FA} + k)$ ,  $\bar{\Psi}_A(k) = \phi_-(-k_{FA} + k)$ ,  $\Psi_B(k) = \phi_-(-k_{FB} + k)$  and  $\bar{\Psi}_B(k) = \phi_-(k_{FB} + k)$ . The field operators are introduced by  $\psi_\nu(x) = \int \frac{dk}{2\pi} \Psi_\nu(k) e^{ikx}$  and  $\bar{\psi}_\nu(x) = \int \frac{dk}{2\pi} \bar{\Psi}_\nu(k) e^{ikx}$ . These field operators are related to lattice operators by

$$C(j) \rightarrow \lambda^A \psi_A(x) e^{ik_{FA}x} + \lambda^{A*} \bar{\psi}_A(x) e^{-ik_{FA}x} + \lambda^B \psi_B(x) e^{-ik_{FB}x} + \lambda^{B*} \bar{\psi}_B(x) e^{ik_{FB}x}, \quad (2)$$

where

$$\lambda^\nu = \begin{bmatrix} i \sin(\theta_\nu/2) \\ \cos(\theta_\nu/2) \end{bmatrix},$$

with  $\theta_A = \theta(k_{FA})$  and  $\theta_B = \theta(-k_{FB})$ .

With polar molecules or atoms loaded on the  $sp$ -ladder, we include all momentum-independent interactions (momentum-dependent part is irrelevant in the Renormalization group flow [39]) allowed by symmetry. The Hamiltonian density of the interactions is given by

$$\mathcal{H}_{\text{int}} = \sum_{\nu\nu'} \frac{1}{2} g_4^{\nu\nu'} [J_\nu J_{\nu'} + \bar{J}_\nu \bar{J}_{\nu'}] + g_2^{\nu\nu'} J_\nu \bar{J}_{\nu'} + g_3 \{ \bar{\psi}_A^* \bar{\psi}_B \psi_A^* \psi_B + \bar{\psi}_B^* \bar{\psi}_A \psi_B^* \psi_A \}, \quad (3)$$

where  $J_\nu =: \psi_\nu^* \psi_\nu$  and  $\bar{J}_\nu =: \bar{\psi}_\nu^* \bar{\psi}_\nu$ . For the symmetric case  $t_s = t_p$ , an Umklapp process

$$\mathcal{H}_{um} = g_u \{ \bar{\psi}_A^* \psi_A \psi_B^* \bar{\psi}_B + \bar{\psi}_B^* \psi_B \psi_A^* \bar{\psi}_A \} \quad (4)$$

becomes allowed for the reason that  $k_{FA} + k_{FB} = \pi$ . Since dipolar interactions between polar molecules or atoms decay as  $1/r^3$ , the leading interaction in the proposed double-well lattice setup [20] is

$$H_{\text{int}} = U \sum_j \left[ a_s^\dagger(j) a_s(j) - \frac{1}{2} \right] \left[ a_p^\dagger(j) a_p(j) - \frac{1}{2} \right].$$

In the weak interacting limit, the  $g$ -ology couplings are related to  $U$  by  $g_4^{\nu\nu} = U$ ,  $g_4^{AB} = g_4^{BA} = U \sin^2(\frac{\theta_A - \theta_B}{2})$ ,  $g_2^{\nu\nu} = U \sin^2(\theta_\nu)$ ,  $g_2^{AB} = g_2^{BA} = U$ ,

$$g_3 = U \sin(\theta_A) \sin(\theta_B),$$

and

$$g_u = U \cos(\theta_A) \cos(\theta_B),$$

at tree level [39]. Considering strong interactions or the finite ranged tail of dipolar interactions, the  $g$ -ology couplings will be renormalized due to neglected irrelevant couplings. By manipulating the direction of dipole moments with an external field, the interaction can be either repulsive or attractive [34–37].

We follow the notation convention of Ref. [40], where the bosonization identity takes the form

$$\begin{aligned} \psi_\nu &= \frac{1}{\sqrt{2\pi}} \eta_\nu e^{-i\sqrt{\pi}(\varphi_\nu + \vartheta_\nu)} \\ \bar{\psi}_\nu &= \frac{1}{\sqrt{2\pi}} \bar{\eta}_\nu e^{i\sqrt{\pi}(\varphi_\nu - \vartheta_\nu)}, \end{aligned} \quad (5)$$

where  $\eta_\nu$  is the Klein factor and  $\vartheta_\nu$  is the dual field of boson field  $\varphi_\nu$ . The charge and orbital boson fields are further introduced here by  $[\varphi_c, \varphi_o] = [\varphi_A, \varphi_B]T$ , with the matrix  $T$  given by

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

and their duals fields are  $[\vartheta_c, \vartheta_o] = [\vartheta_A, \vartheta_B]T$ . The Bosonized Hamiltonian density reads

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_c + \mathcal{H}_o + \mathcal{H}_{\text{mix}}, \\ \mathcal{H}_c &= \frac{u_c}{2} \left[ K_c \Pi_c^2 + \frac{1}{K_c} (\partial_x \varphi_c)^2 \right], \\ \mathcal{H}_o &= \frac{u_o}{2} \left[ K_o \Pi_o^2 + \frac{1}{K_o} (\partial_x \varphi_o)^2 \right] \\ &\quad + \frac{1}{2\pi^2} \left[ g_3 \cos(\sqrt{8\pi} \vartheta_o) + g_u \cos(\sqrt{8\pi} \varphi_o) \right], \\ \mathcal{H}_{\text{mix}} &= u_m \left[ K_m \Pi_c \Pi_o + \frac{1}{K_m} (\partial_x \varphi_c) (\partial_x \varphi_o) \right], \end{aligned} \quad (6)$$

with  $u_{\alpha=c/o} = \sqrt{(v_+ + \tilde{g}_4^{\alpha\alpha}/2\pi)^2 - (\tilde{g}_2^{\alpha\alpha}/2\pi)^2}$ ,  $K_\alpha = \sqrt{\frac{2\pi v_+ + \tilde{g}_4^{\alpha\alpha} - \tilde{g}_2^{\alpha\alpha}}{2\pi v_+ + \tilde{g}_4^{\alpha\alpha} + \tilde{g}_2^{\alpha\alpha}}}$ ,  $u_m = \sqrt{(v_- + \tilde{g}_4^{co}/2\pi)^2 - (\tilde{g}_2^{co}/2\pi)^2}$ ,

and  $K_m = \sqrt{\frac{2\pi v_- + \tilde{g}_4^{co} - \tilde{g}_2^{co}}{2\pi v_- + \tilde{g}_4^{co} + \tilde{g}_2^{co}}}$ , where  $v_+ = (v_{FA} + v_{FB})/2$ ,  $v_- = (-v_{FA} + v_{FB})/2$  and the transformed coupling matrices  $\tilde{g}_4$  and  $\tilde{g}_2$  are given by  $[\tilde{g}] = T^{-1}[g]T$ . The mixing term  $\mathcal{H}_{\text{mix}}$  vanishes for the symmetric case with  $t_s = t_p$ .

*Symmetric case.* For the symmetric case with  $t_s = t_p$  (FIG. 1), the Hamiltonian has an accidental  $Z_2$  symmetry,  $C_j \rightarrow (-1)^j \sigma_x C_j$  and Fermi momenta are related by  $k_{FA} = \pi - k_{FB} \equiv k_F$ . This  $Z_2$  symmetry implies that  $v_{FA} = v_{FB}$ ,  $g_4^{AA} = g_4^{BB}$  and  $g_2^{AA} = g_2^{BB}$ . We find that the transformed coupling matrices  $\tilde{g}_2$  and  $\tilde{g}_4$  are diagonal and that the orbital-charge mixing term  $\mathcal{H}_{\text{mix}}$  vanishes. In other words, the  $Z_2$  symmetry guarantees orbital-charge separation. The charge part  $\mathcal{H}_c$  is quadratic and the orbital part  $\mathcal{H}_o$  is a Sine-Gordon model [41].

With attraction, we have  $K_o < 1$ ,  $g_u > 0$ , and the Sine-Gordon term  $g_u$  is relevant (flows to  $+\infty$ ) in the renormalization group (RG) flow [41]. This corresponds

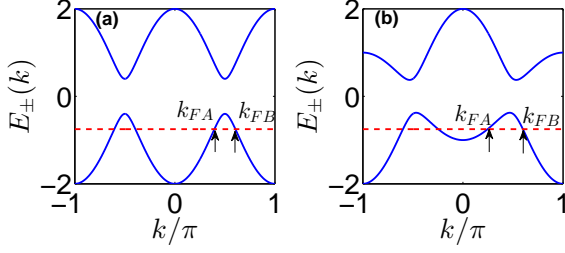


FIG. 1: Sketch of the band structure of the  $sp$ -orbital ladder. Red dashed lines show the level of chemical potentials. (a), the symmetric case with  $t_s = t_p$ . (b), the asymmetric case with  $t_s < t_p$ .

to an orbital gapped phase with  $\cos(\sqrt{8\pi}\varphi_o)$  locked at  $-1$ . In this phase, quantum fluctuations of  $\varphi_o$  become massive, and the divergent susceptibilities are the following: charge density wave (CDW) and PDW [31, 32] given by the operators:

$$\begin{aligned} O_{\text{CDW}}(x) &= \psi_A^* \bar{\psi}_A e^{-2ik_F A x} - \psi_B^* \bar{\psi}_B e^{2ik_F B x} \\ &\propto e^{-2ik_F x} e^{i\sqrt{2\pi}\varphi_c} \sin(\sqrt{2\pi}\varphi_o) \\ O_{\text{PDW}}(x) &= \psi_A \bar{\psi}_B e^{i(k_{FA} + k_{FB})x} + \psi_B \bar{\psi}_A e^{-i(k_{FA} + k_{FB})x} \\ &\propto (-1)^x e^{-i\sqrt{2\pi}\vartheta_c} \sin(\sqrt{2\pi}\varphi_o) \end{aligned}$$

Due to orbital-charge separation, the CDW and PDW correlation functions are readily given by

$$\langle O_{\text{CDW}}(x) O_{\text{CDW}}^\dagger(0) \rangle \propto e^{-2ik_F x} x^{-K_c}, \quad (7)$$

$$\langle O_{\text{PDW}}(x) O_{\text{PDW}}^\dagger(0) \rangle \propto (-1)^x x^{-1/K_c}. \quad (8)$$

Since  $K_c > 1$  for attraction, the algebraic PDW order is dominant. In this phase, the superconducting pairing  $\mathcal{O}_{\text{SC}} = a_s(j)a_p(j)$  oscillates in space with a period of  $\pi$ .

With repulsion, we have  $K_o > 1$ , and thus  $g_3$  is relevant [41]. This gives an orbital gapped phase with  $\cos(\sqrt{8\pi}\vartheta_o)$  locked at 1, because  $g_3 < 0$ . The fluctuations of  $\vartheta_o$  are massive, and the divergent susceptibilities are ODW and superconducting  $\text{SC}^+$  given by the operators:

$$\begin{aligned} O_{\text{ODW}}(x) &= e^{-i(k_{FA} - k_{FB})x} (\psi_A^* \bar{\psi}_B - \psi_B^* \bar{\psi}_A) \\ &\propto e^{-i(k_{FA} - k_{FB})x} e^{i\sqrt{2\pi}\varphi_c} \cos(\sqrt{2\pi}\vartheta_o) \\ O_{\text{SC}^+}(x) &= \psi_A \bar{\psi}_A + \psi_B \bar{\psi}_B \\ &\propto e^{-i\sqrt{2\pi}\vartheta_c} \cos(\sqrt{2\pi}\vartheta_o) \end{aligned}$$

Since  $K_c < 1$  for repulsion, the dominant algebraic order here is ODW, for which the correlation function is given by

$$\langle O_{\text{ODW}}(x) O_{\text{ODW}}^\dagger(0) \rangle \propto e^{-i(k_{FA} - k_{FB})x} x^{-K_c}. \quad (9)$$

In the ODW phase, the particle-hole pairing in terms of lattice operators reads  $\mathcal{O}_{\text{ODW}}(j) = C_j^\dagger \sigma_y C_j$ . This

ODW order is incommensurate with an oscillation period  $2\pi/(k_{FA} - k_{FB})$  in real space. If we go beyond the one-dimensional limit and consider small transverse tunnelings [20], a true long-range ODW order  $\langle O_{\text{ODW}}(x) \rangle \propto e^{i(k_{FA} - k_{FB})x}$  is expected. Such an order breaks time-reversal symmetry.

The ODW and PDW phases predicted by Bosonization analysis are further verified in numerical simulations with matrix products states. The superconducting correlation  $C_{\text{SC}}(j' - j) = \langle a_p^\dagger(j) a_s^\dagger(j) a_s(j') a_p(j') \rangle$  and the orbital density wave correlation  $C_{\text{ODW}}(j' - j) = \langle C_j^\dagger \sigma_y C_j C_{j'}^\dagger \sigma_y C_{j'} \rangle$  are calculated. FIG. 2 shows the Fourier transform of these correlations, defined by  $\mathcal{C}(k) = \sum_{j \neq 0} C(j) e^{-ikj}$ . The sharp peaks of  $\mathcal{C}_{\text{SC}}(k)$  at momenta  $\pm\pi$  on the attractive side tell the quantum state has a PDW order shown in Eq. 8. On the repulsive side sharp dips of  $\mathcal{C}_{\text{ODW}}(k)$  at finite momenta verify the incommensurate ODW order shown in Eq. 9. With numerical calculations, we also find the existence of PDW phase in the strongly attractive regime if  $t_s \neq t_p$ .

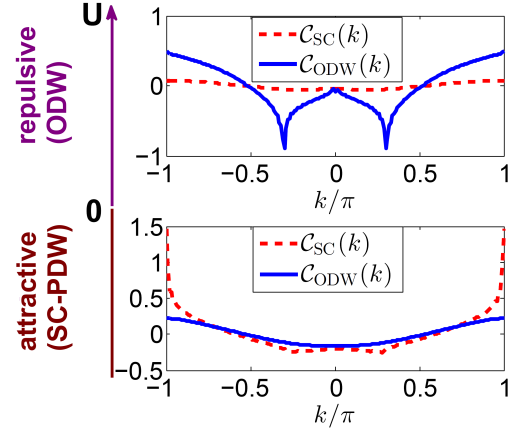


FIG. 2: The phase diagram of the symmetric  $sp$ -orbital ladder with  $t_s = t_p = 2t_{sp}$ .  $\mathcal{C}_{\text{ODW}}(k)$  and  $\mathcal{C}_{\text{SC}}(k)$  show the Fourier transform of the orbital density wave and superconducting correlations, respectively. The numerical results are calculated for a system of 70 particles in a ladder of size 100. In the upper (lower) inset, the interaction  $U = 3t_s$  ( $U = -3t_s$ ).

*Asymmetric case.* For the asymmetric case— $t_s < t_p$  (FIG. 1), the Fermi velocity  $v_{FB} > v_{FA}$  and the orbital-charge separation no longer holds. Thus, the orbital and charge degrees of freedom cannot be treated separately. The other difference with the symmetric case is that the Umklapp process  $g_u$  does not exist. Since the effects of  $g_4$  couplings are just to renormalize the Fermi velocities [42–45]. For simplicity, we do not consider such effects and set  $g_4^{\nu\nu'} = 0$  here. The one-loop RG equations are given

by [45],

$$\begin{aligned}\frac{dg_2^{\nu\nu'}}{dl} &= \frac{g_2^2}{2\pi} \left[ \frac{\delta_{\bar{\nu}\nu'}}{v_+} - \frac{\delta_{\nu\nu'}}{v_{F\bar{\nu}}} \right], \\ \frac{dg_3}{dl} &= \frac{g_3}{2\pi} \sum_{\nu} \left[ \frac{g_2^{\nu\bar{\nu}}}{v_+} - \frac{g_2^{\nu\nu}}{v_{F\nu}} \right],\end{aligned}\quad (10)$$

where  $l$  is the flow parameter ( $l \rightarrow \infty$ ) and  $\bar{\nu} = A$  ( $B$ ) for  $\nu = B$  ( $A$ ). The RG flow of the Sine-Gordon term  $g_3$  is obtained as

$$\begin{aligned}\sqrt{|C|}/g_3(l) \\ = F \left[ -\text{sgn}(g_3 Y) \sqrt{\frac{2|C|D}{\pi v_+}} l + F^{-1}[\sqrt{|C|}/g_3(0)] \right]\end{aligned}\quad (11)$$

with

$$C = \frac{2v_{FA}v_{FB}v_+^2}{v_{FA}v_{FB}+v_+^2} \left[ \frac{g_2^{AB}}{v_+} - \frac{g_2^{AA}}{2v_{FA}} - \frac{g_2^{BB}}{2v_{FB}} \right]^2 - g_3^2, \quad (12)$$

$$D = \frac{v_{FA}v_{FB}+v_+^2}{\pi v_{FA}v_{FB}v_+}, \quad (13)$$

and

$$Y = \frac{g_2^{AB}}{v_+} - \frac{g_2^{AA}}{2v_{FA}} - \frac{g_2^{BB}}{2v_{FB}}. \quad (14)$$

The function  $F$  is the hyperbolic function “sinh” (the trigonometric function “sin”) if  $C > 0$  ( $C < 0$ ). When  $Y > 0$ ,  $g_3$  always flows to  $\infty$  and the system is in some gapped phase. When  $Y < 0$ ,  $g_3$  flows to  $\infty$  only if  $C < 0$ .  $g_3$  is irrelevant only if  $C > 0$  and  $Y < 0$ . In the weak interacting regime,  $Y/U = \frac{1}{v_+} - \frac{\sin^2(\theta_A)}{2v_{FA}} - \frac{\sin^2(\theta_B)}{2v_{FB}}$ . We will consider the regime  $Y/U > 0$  (this condition holds when  $t_{sp}$  is weak compared with  $t_s + t_p$ ) in the following.

With repulsion ( $U > 0$ ,  $Y > 0$ ),  $g_3$  is relevant and flows to  $-\infty$  in RG flow. Then the dual orbital field  $\vartheta_o$  is locked with  $\cos(\sqrt{8\pi}\vartheta_o) = 1$  and its fluctuations  $\vartheta_o$  are massive. The key effect of orbital-charge mixing can be seen from its modification of the dynamics of the conjugate fields, given as

$$\Pi_{\vartheta_o} = \frac{K_o}{u_o} \partial_t \vartheta_o + \frac{K_o u_m}{K_m u_o} \partial_x \varphi_o, \quad (15)$$

$$\Pi_{\varphi_c} = \frac{1}{u_c K_c} \partial_t \varphi_c + \frac{K_m u_m}{K_c u_c} \partial_x \vartheta_o, \quad (16)$$

where  $\Pi_{\varpi}$  is the conjugate field of  $\varpi$ . The Lagrangian is constructed by  $\mathcal{L}(x, t) = \Pi_{\vartheta_o} \partial_t \vartheta_o + \Pi_{\varphi_c} \partial_t \varphi_c - \mathcal{H}$ . With massive fluctuations of  $\vartheta_o$  integrated out, the Lagrangian of the charge field  $\varphi_c$  is given by

$$\mathcal{L}_c = \frac{1}{2\gamma} \left[ \frac{1}{u} (\partial_t \varphi_c)^2 - u (\partial_x \varphi_c)^2 \right] + O((\partial \varphi_c)^4), \quad (17)$$

with the renormalized Luttinger parameter and sound velocity given by

$$\gamma = \frac{K_c}{\sqrt{1 - \frac{K_c K_o}{K_m^2} \frac{u_m^2}{u_c u_o}}}, \quad (18)$$

$$u = \sqrt{u_c^2 - u_m^2 \frac{u_c K_c K_o}{u_o K_m^2}}. \quad (19)$$

To zeroth order in the interaction  $U$ , the renormalized Luttinger parameter is  $\gamma = \left[ 1 - \left( \frac{v_-}{v_+} \right)^2 \right]^{-1/2}$ . Our result reproduces the perturbative result [45] when the orbital-charge mixing term is small. The diverging susceptibilities are ODW and  $SC^+$ , and the corresponding correlation functions are given as

$$\langle O_{SC^+}(x) O_{SC^+}^\dagger(0) \rangle \propto x^{-1/\gamma}, \quad (20)$$

$$\langle O_{ODW}(x) O_{ODW}^\dagger(0) \rangle \propto e^{-i(k_{FA} - k_{FB})x} x^{-\gamma}. \quad (21)$$

With sufficiently weak repulsion  $\gamma > 1$ , the dominant order is  $SC^+$ , of which the pairing in terms of lattice operators is  $O_{SC} = a_s(j)a_p(j)$ . We emphasize here that this pairing does not oscillate in real space. Such a superconducting phase arises in the repulsive regime due to the orbital-charge mixing and the pinning effect of the dual orbital field  $\vartheta_o$ . The Sine-Gordon term  $g_3$  causing this pinning effect is finite only when the coupling of  $sp$ -orbitals  $t_{sp}$  is finite, and  $g_3$  is monotonically increasing when  $t_{sp}$  is increased. Thus the transition temperature of this repulsive superconducting phase can be increased by tuning  $t_{sp}$ , which makes this exotic superconducting phase potentially realizable in experiments. With stronger repulsion, the renormalized Luttinger parameter  $\gamma$  decreases. Eventually with repulsion larger than some critical strength, we have  $\gamma < 1$ , and the repulsive superconducting phase gives way to the ODW phase. Our numerical results cannot distinguish the ODW and SC phases in the weak interacting regime due to computational complexity [46].

With attractive interaction, the condition  $Y/U > 0$  gives  $Y < 0$ . Thus  $g_3$  is relevant and flows to  $+\infty$  when  $C < 0$ . The Sine-Gordon term  $\cos(\sqrt{8\pi}\vartheta_o)$  is locked at  $-1$ , and the dominant order is superconducting  $SC^-$ , given by

$$\begin{aligned}O_{SC^-} &= \psi_A \bar{\psi}_A - \psi_B \bar{\psi}_B \\ &\propto e^{-i\sqrt{2\pi}\vartheta_c} \sin(\sqrt{2\pi}\vartheta_o).\end{aligned}\quad (22)$$

In numerical simulations we find the  $SC^-$  phase competing with PDW in the strongly attractive regime. When  $g_3$  is irrelevant ( $C > 0$ ,  $Y < 0$ ), the orbital ladder is in a two component Luttinger liquid phase exhibiting two gapless normal modes and each mode is a mixture of orbital and charge.

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- [1] M. A. Lewenstein and W. V. Liu, *Nature Physics* **7**, 101 (2011).
- [2] Y. Tokura and N. Nagaosa, *Science* **288**, 462 (2000).
- [3] A. Isacsson and S. M. Girvin, *Phys. Rev. A* **72**, 053604 (2005).
- [4] W. V. Liu and C. Wu, *Phys. Rev. A* **74**, 013607 (2006).
- [5] A. B. Kuklov, *Phys. Rev. Lett.* **97**, 110405 (2006).
- [6] L.-K. Lim, C. M. Smith, and A. Hemmerich, *Phys. Rev. Lett.* **100**, 130402 (2008).
- [7] X. Li, Z. Zhang, and W. V. Liu, *Phys. Rev. Lett.* **108**, 175302 (2012).
- [8] G. Wirth, M. Ölschläger, and A. Hemmerich, *Nature Physics* **7**, 147 (2011).
- [9] P. Soltan-Panahi, D.-S. Lühmann, J. Struck, P. Windpassinger, and K. Sengstock, *Nature Physics* **8**, 71 (2012).
- [10] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
- [11] M. Lewenstein, A. Sanpera, and V. Ahufinger, *Ultracold atoms in optical lattices: Simulating Quantum Many-body systems* (Oxford University Press, Oxford, 2012).
- [12] T. Rom, T. Best, D. van Oosten, U. Schneider, S. Folling, B. Paredes, and I. Bloch, *Nature* **444**, 733 (2006).
- [13] A. Moreo and D. J. Scalapino, *Phys. Rev. Lett.* **98**, 216402 (2007).
- [14] A. F. Ho, M. A. Cazalilla, and T. Giamarchi, *Phys. Rev. A* **79**, 033620 (2009).
- [15] Y.-J. Lin, R. L. Compton, K. Jimenez-Garcia, J. V. Porto, and I. B. Spielman, *Nature* **462**, 628 (2009).
- [16] P. Wang, Z.-Q. Yu, Z. Fu, J. Miao, L. Huang, S. Chai, H. Zhai, and J. Zhang, *Phys. Rev. Lett.* **109**, 095301 (2012).
- [17] L. W. Cheuk, A. T. Sommer, Z. Hadzibabic, T. Yefsah, W. S. Bakr, and M. W. Zwierlein, *Phys. Rev. Lett.* **109**, 095302 (2012).
- [18] L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, *Nature* **483**, 302 (2012).
- [19] M. Ölschläger, G. Wirth, T. Kock, and A. Hemmerich, *Phys. Rev. Lett.* **108**, 075302 (2012).
- [20] X. Li, E. Zhao, and W. V. Liu, *ArXiv e-prints* (2012), 1205.0254.
- [21] E. Zhao and W. V. Liu, *Phys. Rev. Lett.* **100**, 160403 (2008).
- [22] C. Wu, *Phys. Rev. Lett.* **100**, 200406 (2008).
- [23] V. M. Stojanović *et al*, *Phys. Rev. Lett.* **101**, 125301 (2008).
- [24] C. Wu, *Mod. Phys. Lett. B* **23**, 1 (2009).
- [25] H.-H. Hung, W.-C. Lee, and C. Wu, *Phys. Rev. B* **83**, 144506 (2011).
- [26] Z. Cai, Y. Wang, and C. Wu, *Phys. Rev. A* **83**, 063621 (2011).
- [27] Q. Zhou, J. V. Porto, and S. Das Sarma, *Phys. Rev. B* **83**, 195106 (2011).
- [28] K. Sun, W. V. Liu, A. Hemmerich, and S. Das Sarma, *Nature Phys.* pp. 67–70 (2012).
- [29] H.-Y. Hui, R. Barnett, J. V. Porto, and S. Das Sarma, *ArXiv e-prints* (2012), 1208.6300.
- [30] Z. Zhang *et al*, *Phys. Rev. A* **82**, 033610 (2010).
- [31] E. Berg, E. Fradkin, and S. A. Kivelson, *Nat Phys* **5**, 830 (2009).
- [32] A. Jaefari and E. Fradkin, *Phys. Rev. B* **85**, 035104 (2012).
- [33] R. Wei and E. J. Mueller, *Phys. Rev. Lett.* **108**, 245301 (2012).
- [34] K.-K. Ni, S. Ospelkaus, M. H. G. de Miranda, A. Pe'er, B. Neyenhuis, J. J. Zirbel, S. Kotochigova, P. S. Julienne, D. S. Jin, and J. Ye, *Science* **322**, 231 (2008).
- [35] M. H. G. de Miranda, A. Chotia, B. Neyenhuis, D. Wang, G. Quemener, S. Ospelkaus, J. L. Bohn, J. Ye, and D. S. Jin, *Nat Phys* **7**, 502 (2011).
- [36] A. Chotia, B. Neyenhuis, S. A. Moses, B. Yan, J. P. Covey, M. Foss-Feig, A. M. Rey, D. S. Jin, and J. Ye, *Phys. Rev. Lett.* **108**, 080405 (2012).
- [37] C.-H. Wu, J. W. Park, P. Ahmadi, S. Will, and M. W. Zwierlein, *Phys. Rev. Lett.* **109**, 085301 (2012).
- [38] M. Lu, S. H. Youn, and B. L. Lev, *Phys. Rev. Lett.* **104**, 063001 (2010).
- [39] R. Shankar, *Rev. Mod. Phys.* **66**, 129 (1994).
- [40] D. Sénéchal, eprint *arXiv:cond-mat/9908262* (1999), *arXiv:cond-mat/9908262*.
- [41] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford, Oxford, 2003).
- [42] M. Fabrizio, *Phys. Rev. B* **48**, 15838 (1993).
- [43] L. Balents and M. P. A. Fisher, *Phys. Rev. B* **53**, 12133 (1996).
- [44] H.-H. Lin, L. Balents, and M. P. A. Fisher, *Phys. Rev. B* **56**, 6569 (1997).
- [45] U. Ledermann and K. Le Hur, *Phys. Rev. B* **61**, 2497 (2000).
- [46] N. Schuch, I. Cirac, and F. Verstraete, *Phys. Rev. Lett.* **100**, 250501 (2008).